MATH 20D Spring 2023 Lecture 24.

Systems in Normal Form, Eigenvalue, and Eigenvectors.

- CAPE course and professor evaluations are available. Please fill this out BEFORE 8am on June 10th.

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- Midterm 2 grades are available, regrade request window closing tonight.
- HW 4 grades available, regrade request window closing Sunday 11:59pm.

Outline

Contents

We saw **two first order systems of differential equations**

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(i) **Mixing Involving Two Interconnected Tanks:**

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$$y'(t) = \frac{1}{3}x(t) - \frac{1}{3}y(t)$$

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$x(t)$ and $y(t)$ give the masses of salt in tanks A and B respectively.

(ii) **Higher order ODE's give first order system of ODE's**

Let $x_1(t) = y(t)$, $x_2(t) = y'(t)$, and $x_3(t) = y''(t)$. Then the third order ODE

$$y'''(t) + 2y''(t) + y'(t) = 0 \quad (1)$$

is equivalent to the first order system of ODE's

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= x_3(t) \\ x_3'(t) &= -x_2(t) - 2x_3(t) \end{aligned}$$

Last Time II

- Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix}$.

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- If we introduce the **vector valued function** $\mathbf{x}(t) = \mathrm{col}(x_1(t), x_2(t), x_3(t))$ then the system can be written in **matrix notation**

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

Example

Rewrite each of the systems below as a first order equation in matrix notation

(a) $y'' + 2y' + 3y = \frac{1}{t}$ **(b)** $ty'' + (1-t)y = e^t$ **(c)** $\begin{aligned} 2x'' + 6x - 2y &= 0 \\ y'' + 2y - 2x &= 0 \end{aligned}$

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Definition

A system of differential equation is in ***normal form*** *if it is expressed as*

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{2}$$

where $\mathbf{x}(t) = \mathrm{col}(x_1(t), \ldots, x_n(t))$ *and* $\mathbf{f}(t) = \mathrm{col}(f_1(t), \ldots, f_n(t))$.

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- *We say that (2) is **homogeneous** if* $\mathbf{f}(t) \equiv 0$.
- *Say that (2) has **constant coefficients** if* $A(t) = A$ *has constant entries.*

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Leading Questions

Suppose A is a 2×2 matrix with constant entries.

- How do we write down a general solutions to the equation

$$\mathbf{x}'(t) = A\mathbf{x}(t)?$$

where $\mathbf{x}(t) = \mathrm{col}(x_1(t), x_2(t))$.

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If $a \neq 0$, b, and c are constant then the IVP

$$ay'' + by' + cy = 0, \qquad y(0) = Y_0, \quad y'(0) = Y_1$$

is given in matrix notation as $\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}\mathbf{x}(t)$, $\mathbf{x}(0) = \mathrm{col}(Y_0, Y_1)$.

Contents

Defining Eigenvalues and Eigenvectors

- Let $n \geqslant 1$ and suppose A is an n-by-n matrix with constant entries.

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- Write
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so that multiplication by A gives a function $\mathbf{v} \mapsto A\mathbf{v}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$.

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Example

Show that $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$ with eigenvalue $\lambda = 1$.

Another Example

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- Show that

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is an eigenvector of $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ with eigenvalue $\lambda = 3$.

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- Show that
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Finding Eigenvalues and Eigenvectors

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- The above equation admits a non-zero solution if and only if $A - \lambda I$ is **not** invertible.
- Recall that a 2-by-2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is **invertible** if and only if the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is non-zero.

Summary

- Given a 2-by-2 matrix A. We can solve for the eigenvalues of A by solving the **characteristic equation**

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Example

Find the eigenvalues and eigenvectors of the matrices below

(a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$